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Trigonometric Yang–Baxterization of coloured $\check{R}$-matrix

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Abstract. For a given coloured solution $\check{R}(\lambda, \mu)$ of the (quantum) Yang-Baxter equation in appropriate form, we present an explicit prescription to generate trigonometric solutions $\check{R}(\lambda, \mu; x)$ of the Yang-Baxter equation with spectral parameters x. When the given $\check{R}(\lambda, \mu)$ admits the Birman-Wenzl algebraic structure with colour, we find that two types of the $\check{R}(\lambda, \mu; x)$ are generated by such $\check{R}(\lambda, \mu)$. New explicit examples associated with the fundamental representations of A_n, B_n, C_n and D_n are given in terms of this prescription.

1. Introduction

It is well known that the (quantum) Yang-Baxter equation (YBE) plays a fundamental role in the theory of (1+1)- or 2-dimensional integrable quantum systems [1, 2], including lattice statistical models and nonlinear field theory (for review, see [4]), and is also closely related to some other fields, such as the quantum group (QP) [5, 6], knot theory (for the relationship between statistical models and knot theory, see [3]) and conformal field theory [7], etc, in both mathematics and physics.

A solution of the YBE

$$\check{R}_{12}(x)\check{R}_{23}(xy)\check{R}_{12}(y)=\check{R}_{23}(y)\check{R}_{12}(xy)\check{R}_{23}(x) \tag{1.1}$$

contains two parameters. One is $q=e^h$, where h is the Planck constant, and the other is the spectral parameter $x=e^{-u}$. Many solutions of the YBE (1.1) have been constructed systematically in terms of the following methods: one is that the $\check{R}(x)$-matrix can be generated from the classical R-matrix based on the QP [6]; another is the explicit prescription for Yang-Baxterization by starting from the braid group representation (BGR) [8, 9].

On the other hand, another form of the YBE can be written as follows

$$\check{R}_{12}(\lambda, \mu)\check{R}_{23}(\lambda, \nu)\check{R}_{12}(\mu, \nu)=\check{R}_{23}(\mu, \nu)\check{R}_{12}(\lambda, \nu)\check{R}_{23}(\lambda, \mu) \tag{1.2}$$

where $\check{R}(\lambda, \mu)$ is a matrix acting on the tensor space $V(\lambda)\otimes V(\mu)$. We need only consider the case that $\check{R}(\lambda, \mu)$ is not equal to $\check{R}(\lambda\nu^{-1})$, because the YBE (1.2) is reduced to the YBE (1.1) by taking $x=\lambda\mu^{-1}$ and $y=\mu\nu^{-1}$ when $\check{R}(\lambda, \mu)=\check{R}(\lambda\mu^{-1})$. Recently, Murakami [10] has found a (4×4)-dimensional solution of the YBE (1.2) and constructed a multivariable Alexander polynomial in terms of this solution. The parameters λ, μ and ν appearing in the YBE (1.2) are interpreted as colours [10]. In our previous

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work [11] we have found two types of solution associated with the fundamental representation of A_1, based on weight conservation. The solution given by Murakami is a special case of the latter solution in [11]. The $\check{R}(\lambda, \mu)$-matrices associated with the fundamental representations of A_n $(n>1)$, B_n, C_n and D_n have been obtained, and the Birman–Wenzl (BW) algebra with colour has been constructed in [12]. It is shown that the $\check{R}(\lambda, \mu)$-matrices for B_n, C_n and D_n admit this algebraic structure.

The most general YBE has the following form:

$$\check{R}_{12}(\lambda, \mu; x)\check{R}_{23}(\lambda, \nu; xy)\check{R}_{12}(\mu, \nu; y)=\check{R}_{23}(\mu, \nu; y)\check{R}_{12}(\lambda, \nu; xy)\check{R}_{23}(\lambda, \mu; x). \qquad (1.3)$$

The YBE (1.1) and the YBE (1.2) can be regarded as the limits of the YBE (1.3) by taking $\lambda=\mu=\nu$ and $x=y=0$, respectively.

In consideration of the difficulty solving the YBE (1.3) directly, in this paper we construct an explicit prescription, called Yang–Baxterization of coloured $\check{R}$-matrix, to generate the $\check{R}(\lambda, \mu; x)$-matrix from a given $\check{R}(\lambda, \mu)$-matrix in appropriate form. In fact, this prescription is the generalization of the prescription for Yang–Baxterization in [8].

In [8], the $\check{R}(x)$ can be represented by the BGR and the unit matrix. The form of the $\check{R}(x)$ depends on the reduction relation satisfied by the BGR. For instance, the $\check{R}(x)$-matrices corresponding to the BGR S with two and three distinct eigenvalues have the following forms

$$\check{R}(x)=\lambda_1 x S^{-1}+\lambda_2^{-1}S \qquad (1.4)$$

and

$$\check{R}(x)=\lambda_1 x(x-1)S^{-1}+\lambda_2^{-1}\lambda_3^{-1}(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)I-\lambda_3^{-1}(x-1)S \qquad (1.5)$$

respectively. In the latter formula, S needs to satisfy the condition (3.27) given in [8]. The correctness of this condition can be proved when S admits the BW algebraic structure [9].

In this paper we start from $\check{R}(\lambda, \mu)$ instead of S. In terms of the $\check{R}(\lambda, \mu)$-matrix in appropriate form, we construct the explicit forms of the $\check{R}(\lambda, \mu; x)$-matrix, which are similar to the formulae (1.4) and (1.5). We find that two solutions of the YBE (1.3) are generated from one $\check{R}(\lambda, \mu)$ satisfying the BW algebra with colour. The new $\check{R}(\lambda, \mu; x)$-matrices are obtained by starting from $\check{R}(\lambda, \mu)$ associated with the fundamental representations of A_n, B_n, C_n and D_n.

This paper is organized as follows. We first discuss the prescription for Yang–Baxterization of coloured $\check{R}$-matrix in section 2. Then the relationship between this prescription and the BW algebra with colour is shown in section 3. In section 4, the $\check{R}(\lambda, \mu; x)$-matrices for A_n, B_n, C_n and D_n are given as examples.

2. Yang–Baxterization of coloured $\check{R}$-matrix

Let us first discuss some properties of $\check{R}(\lambda, \mu)$. If the inverse $\check{R}^{-1}(\lambda, \mu)$ of $\check{R}(\lambda, \mu)$ exist, the YBE (1.2) leads to five equivalent equations

$$\begin{aligned}
&\check{R}_{12}(\lambda, \mu)\check{R}_{23}(\lambda, \nu)\check{R}_{12}^{-1}(\nu, \mu)=\check{R}_{23}^{-1}(\nu, \mu)R_{12}(\lambda, \nu)\check{R}_{23}(\lambda, \mu)\\
&\check{R}_{12}^{-1}(\mu, \lambda)\check{R}_{23}(\lambda, \nu)\check{R}_{12}(\mu, \nu)=\check{R}_{23}(\mu, \nu)\check{R}_{12}(\lambda, \nu)\check{R}_{23}^{-1}(\nu, \lambda)\\
&\check{R}_{12}(\lambda, \mu)\check{R}_{23}^{-1}(\nu, \lambda)\check{R}_{12}^{-1}(\nu, \mu)=\check{R}_{23}^{-1}(\nu, \mu)\check{R}_{12}^{-1}(\nu, \lambda)\check{R}_{23}(\lambda, \mu)\\
&\check{R}_{12}^{-1}(\mu, \lambda)\check{R}_{23}^{-1}(\nu, \lambda)\check{R}_{12}(\mu, \nu)=\check{R}_{23}(\mu, \nu)\check{R}_{12}^{-1}(\nu, \lambda)\check{R}_{23}^{-1}(\mu, \lambda)\\
&\check{R}_{12}^{-1}(\mu, \lambda)\check{R}_{23}^{-1}(\nu, \lambda)\check{R}_{12}^{-1}(\nu, \mu)=\check{R}_{23}^{-1}(\nu, \mu)\check{R}_{12}^{-1}(\nu, \lambda)\check{R}_{23}^{-1}(\mu, \lambda).
\end{aligned} \qquad (2.1)$$

Besides $\check{R}(\lambda, \mu)$ we introduce the matrix $I(\lambda, \mu)$ satisfying the following relations:

$$\begin{aligned} &I(\lambda, \mu)I(\mu, \lambda) = I(\lambda, \lambda) = I(\mu, \mu) = I \\ &I_{12}(\lambda, \mu)I_{23}(\lambda, \nu)I_{12}(\mu, \nu) = I_{23}(\mu, \nu)I_{12}(\lambda, \nu)I_{23}(\lambda, \nu) \end{aligned} \tag{2.2}$$

where I is the unit matrix.

In order to construct $\check{R}(\lambda, \mu; x)$, we need other relations satisfied by $\check{R}(\lambda, \mu)$ besides equations (2.1) and (2.2). For simplicity, we will restrict our discussion to the $\check{R}(\lambda, \lambda)$-matrix satisfying the following relations

$$\check{R}(\lambda, \lambda) = (\Lambda_1 + \Lambda_2)I - \Lambda_1\Lambda_2\check{R}^{-1}(\lambda, \lambda) \tag{2.3}$$

or

$$\check{R}(\lambda, \lambda)\check{R}(\lambda, \lambda) = \left(\sum_{i=1}^{3} \Lambda_i\right)\check{R}(\lambda, \lambda) - \left(\sum_{i<j}^{3} \Lambda_i\Lambda_j\right)I + \left(\prod_{i=1}^{3}\right)\check{R}^{-1}(\lambda, \lambda) \tag{2.4}$$

where Λ_i ($i = 1, 2$ or 1, 2, 3) are the distinct eigenvalues of the $\check{R}(\lambda, \lambda)$-matrix.

By generalizing the boundary condition, the initial condition and the unitary condition in [8], we assume that the $\check{R}(\lambda, \mu; x)$-matrix satisfies the conditions:

(1) the boundary condition

$$\check{R}(\lambda, \mu; 0) \propto \check{R}(\lambda, \mu) \qquad \check{R}(\lambda, \mu; \infty) \propto \check{R}^{-1}(\mu, \lambda) \tag{2.5}$$

(2) the initial condition

$$\check{R}(\lambda, \mu; 1) \propto I(\lambda, \mu) \tag{2.6}$$

(3) the unitarity condition:

$$\check{R}(\lambda, \mu; x)\check{R}(\mu, \lambda; x^{-1}) \propto I. \tag{2.7}$$

In the following we construct the explicit forms of $\check{R}(\lambda, \mu; x)$. Because these forms depend on the relations (2.3) and (2.4), we will discuss them in two cases.

Case (i). In the case of equation (2.3) satisfied by $\check{R}(\lambda, \lambda)$, we assume that $\check{R}(\lambda, \mu)$ satisfies the relation

$$\check{R}(\lambda, \mu) = \beta(\lambda, \mu)I(\lambda, \mu) - \alpha(\lambda, \mu)\check{R}^{-1}(\mu, \lambda) \tag{2.8}$$

where

$$\beta(\lambda, \lambda) = \Lambda_1 + \Lambda_2 \qquad \alpha(\lambda, \lambda) = \Lambda_1\Lambda_2 \tag{2.9}$$

and we construct $\check{R}(\lambda, \mu; x)$ in the form

$$\check{R}(\lambda, \nu; x) = f_1(\lambda, \mu)x\check{R}^{-1}(\mu, \lambda) + f_2(\lambda, \mu)\check{R}(\lambda, \mu) \tag{2.10}$$

where $f_i(\lambda, \mu)(i = 1, 2)$ are the determined parameters. It is obvious that equation (2.10) satisfies the boundary condition equation (2.5). By using equation (2.8) the initial condition equation (2.6) gives

$$f_1(\lambda, \mu) - \alpha(\lambda, \mu)f_2(\lambda, \mu) = 0. \tag{2.11}$$

It is easy to verify that equation (2.10) satisfies the unitarity condition equation (2.7) in terms of equations (2.8) and (2.11). By taking $f_2 = f$, $f(\lambda, \mu)$ is an arbitrary scalar function, we rewrite equation (2.12) as follows

$$\check{R}(\lambda, \mu; x) = f(\lambda, \mu)(\alpha(\lambda, \mu)x\check{R}^{-1}(\mu, \lambda) - \check{R}(\lambda, \mu)). \tag{2.12}$$

Substituting equation (2.12) into the YBE (1.3), we find that $\check{R}(\lambda, \mu)$ should be satisfied by the relation

$$\alpha(\lambda, \mu)\alpha(\mu, \nu)(\check{R}_{12}^{-1}(\mu, \lambda)\check{R}_{23}(\lambda, \nu)\check{R}_{12}^{-1}(\nu, \mu) - \check{R}_{23}^{-1}(\nu, \mu)\check{R}_{12}(\lambda, \nu)\check{R}_{23}^{-1}(\mu, \lambda))$$
$$+ \alpha(\lambda, \nu)(\check{R}_{12}(\lambda, \mu)\check{R}_{23}^{-1}(\nu, \lambda)\check{R}_{12}(\mu, \nu)$$
$$- \check{R}_{23}(\mu, \nu)\check{R}_{12}^{-1}(\nu, \lambda)\check{R}_{23}(\lambda, \nu)) = 0 \tag{2.13}$$

where equations (1.2) and (2.1) have been used. By substituting equation (2.8) into the latter equation in (2.2) and using the YBE (1.2) and equation (2.1), it is not difficult to prove that equation (2.13) is an identity.

Case (ii). The case of the $\check{R}(\lambda, \lambda)$ having the three distinct eigenvalues. For simplicity, we only consider the case that the eigenvalues of $\check{R}(\lambda, \lambda)$ is independent of λ. We assume that $\check{R}(\lambda, \mu)$ and $I(\lambda, \mu)$ satisfy the following relations:

$$\begin{aligned} &\check{R}_{12}(\lambda, \mu)I_{23}(\lambda, \nu)I_{12}(\mu, \nu) = I_{12}(\lambda, \mu)I_{23}(\lambda, \nu)\check{R}_{12}(\mu, \nu) \\ &I_{23}(\mu, \nu)I_{12}(\lambda, \nu)\check{R}_{23}(\lambda, \mu) = \check{R}_{23}(\mu, \nu)I_{12}(\lambda, \nu)I_{23}(\lambda, \mu). \end{aligned} \tag{2.14}$$

From equations (2.2) and (2.14) it is easy to derive

$$I(\lambda, \mu)\check{R}(\mu, \lambda) = \check{R}(\lambda, \mu)I(\mu, \lambda) \tag{2.15}$$

and

$$\begin{aligned} &\check{R}_{12}(\lambda, \mu)I_{23}(\lambda, \nu)I_{12}(\mu, \nu) = I_{23}(\mu, \nu)\check{R}_{12}(\lambda, \nu)I_{23}(\lambda, \mu) \\ &I_{23}(\mu, \nu)I_{12}(\lambda, \nu)\check{R}_{23}(\lambda, \mu) = I_{12}(\lambda, \mu)\check{R}_{23}(\lambda, \nu)I_{12}(\mu, \nu) \\ &\check{R}_{12}(\lambda, \mu)\check{R}_{23}(\lambda, \nu)I_{12}(\mu, \nu) = I_{23}(\mu, \nu)\check{R}_{12}(\lambda, \nu)\check{R}_{23}(\lambda, \mu) \\ &I_{12}(\lambda, \mu)\check{R}_{23}(\lambda, \nu)\check{R}_{12}(\mu, \nu) = \check{R}_{23}(\mu, \nu)\check{R}_{12}(\lambda, \nu)I_{23}(\lambda, \mu) \\ &\check{R}_{12}(\lambda, \mu)I_{23}(\lambda, \nu)\check{R}_{12}(\mu, \nu) = \check{R}_{12}(\lambda, mu)\check{R}_{12}(\mu, \lambda)I_{23}(\mu, \nu)I_{12}(\lambda, \nu)I_{23}(\lambda, \mu) \\ &\check{R}_{23}(\mu, \nu)I_{12}(\lambda, \nu)\check{R}_{23}(\lambda, \mu) = I_{12}(\lambda, \mu)I_{23}(\lambda, \nu)I_{12}(\mu, \nu)\check{R}_{23}(\mu, \lambda)\check{R}_{23}(\lambda, \mu). \end{aligned} \tag{2.16}$$

The use of equations (2.4), (2.2) and (2.14) give rise to

$$\check{R}(\lambda, \mu)\check{R}(\mu, \lambda) = \left(\sum_{i=1}^{3} \Lambda_i\right)\check{R}(\lambda, \mu)I(\mu, \lambda) - \left(\sum_{i<j}^{3} \Lambda_i\Lambda_j\right)I + \left(\prod_{i=1}^{3} \Lambda_i\right)\check{R}^{-1}(\mu, \lambda)I(\mu, \lambda). \tag{2.17}$$

Consider the boundary condition equation (2.5) and the initial condition equation (2.6). We construct $\check{R}(\lambda, \mu; x)$ in the form

$$\check{R}(\lambda, \mu; x) = Ax(x-1)\check{R}^{-1}(\mu, \lambda) + BxI(\lambda, \mu) + C(x-1)\check{R}(\lambda, \mu). \tag{2.18}$$

From the unitarity condition equation (2.7) the relationship among the determined coefficients A, B and C can be described in three ways:

Case (a) $\quad C = -\Lambda_1^{-1}\Lambda_3^{-1}A$

$$B = \left(\prod_{i=1}^{3} \Lambda_i^{-1}\right)(\Lambda_1 + \Lambda_2)(\Lambda_2 + \Lambda_3)A \tag{2.19}$$

Case (b) $\quad C = -\Lambda_2^{-1}\Lambda_3^{-1}A$

$$B = \left(\prod_{i=1}^{3} \Lambda_i^{-1}\right)(\Lambda_1 + \Lambda_2)(\Lambda_1 + \Lambda_3)A \tag{2.20}$$

Case (c) $\quad C = -\Lambda_1^{-1}\Lambda_2^{-1}A$

$$B = \left(\prod_{i=1}^{3} \Lambda_i^{-1}\right)(\Lambda_1 + \Lambda_3)(\Lambda_2 + \Lambda_3)A. \tag{2.21}$$

In the derivation of equations (2.19)–(2.21), equations (2.15) and (2.17) have been used. It is easy to see that case (b) and case (c) can be obtained by exchanging, respectively, $\Lambda_1 \leftrightarrow \Lambda_2$ and $\Lambda_2 \leftrightarrow \Lambda_3$ from case (a). So we only need to consider case (a).

Under case (a), by choosing $A = \Lambda_1$, equation (2.18) reads

$$\check{R}(\lambda, \mu; x) = \Lambda_1 x(x-1)\check{R}^{-1}(\mu, \lambda) + QxI(\lambda, \mu) - \Lambda_3^{-1}(x-1)\check{R}(\lambda, \mu) \tag{2.22}$$

where

$$Q = \Lambda_2^{-1}\Lambda_3^{-1}(\Lambda_1 + \Lambda_2)(\Lambda_2 + \Lambda_3). \tag{2.23}$$

Substituting equation (2.22) into the YBE (1.3) and using equations (1.2), (2.1), (2.2) and (2.14)–(2.17), after a lengthy calculation we derive the condition for $\check{R}(\lambda, \mu; x)$ to satisfy the YBE (1.3) as follows:

$$\Lambda_3^{-1}F_1(\check{R}) - \Lambda_1 F_2(\check{R}) - Q(F_3(\check{R}) - \Lambda_2^{-1}F_4(\check{R}) + \Lambda_2 F_5(\check{R})) = 0 \tag{2.24}$$

where

$$\begin{aligned}
F_1(\check{R}) &= \check{R}_{12}(\lambda, \mu)\check{R}_{23}^{-1}(\nu, \lambda)\check{R}_{12}(\mu, \nu) - \check{R}_{23}(\mu, \nu)\check{R}_{12}^{-1}(\nu, \lambda)\check{R}_{23}(\lambda, \mu) \\
F_2(\check{R}) &= \check{R}_{12}^{-1}(\mu, \lambda)\check{R}_{23}(\lambda, \nu)\check{R}_{12}^{-1}(\nu, \mu) - \check{R}_{23}^{-1}(\nu, \mu)\check{R}_{12}(\lambda, \nu)\check{R}_{23}^{-1}(\mu, \lambda) \\
F_3(\check{R}) &= (\check{R}_{12}(\lambda, \mu)\check{R}_{23}^{-1}(\nu, \lambda) - \check{R}_{12}^{-1}(\mu, \lambda)\check{R}_{23}(\lambda, \nu))I_{12}(\mu, \nu) \\
&\quad + I_{12}(\lambda, \mu)(\check{R}_{23}^{-1}(\nu, \lambda)\check{R}_{12}(\mu, \nu) - \check{R}_{23}(\lambda, \nu)\check{R}_{12}^{-1}(\nu, \mu)) \\
F_4(\check{R}) &= \check{R}_{12}(\lambda, \mu)I_{23}(\lambda, \mu)I_{12}(\mu, \nu) - I_{23}(\mu, \nu)I_{12}(\lambda, \nu)\check{R}_{23}(\lambda, \mu) \\
F_5(\check{R}) &= \check{R}_{12}^{-1}(\mu, \lambda)I_{23}(\lambda, \nu)I_{12}(\lambda, \nu) - I_{23}(\mu, \nu)I_{12}(\lambda, \nu)\check{R}_{23}^{-1}(\mu, \lambda).
\end{aligned} \tag{2.25}$$

In general we cannot prove directly the condition (2.23) based on the YBE (1.2), equations (2.2) and (2.14). However, we are able to prove its validity when $\check{R}(\lambda, \mu)$ admits the BW algebraic structure with colour. This will be shown in section 3. If one does not know whether the starting $\check{R}(\lambda, \mu)$ satisfies the BW algebra with colour or not, then one needs to check equation (2.23) for $\check{R}(\lambda, \mu)$ or the YBE (1.3) for $\check{R}(\lambda, \mu; x)$ given by equation (2.21).

3. Relationship between the BW algebra with colour and the Yang–Baxterization

It has been shown in section 2 that the correctness of the formula (2.21) depends on the condition (2.23). In this section, we will prove that equation (2.23) is the identity if $\check{R}(\lambda, \mu)$ satisfies the BW algebra with colour.

The BW algebra with colour is generated by the operators $G_i(\lambda, \mu)$, $I_i(\lambda, \mu)$ and $E_i(\lambda, \mu)$, and depends on two parameters m and l which are independent of the colours [12]. The algebraic relations among these operators have been given by equations (4.1)–(4.4) in [12]. The linear representations $A_i(\lambda, \mu)$ (stand for $G_i(\lambda, \mu)$, $I_i(\lambda, \mu)$ and $E_i(\lambda, \mu)$) on the tensor space $\otimes_{i=1}^{n} V(\lambda_i)$ can be written in the form

$$A_i(\lambda, \mu) = I(\lambda_1)\otimes\cdots\otimes I(\lambda_{i-1})\otimes A(\lambda, \mu)\otimes I(\lambda_{i+2})\otimes\cdots\otimes I(\lambda_n) \tag{3.1}$$

where $A(\lambda, \mu)$ is a matrix: $V(\lambda)\otimes V(\mu) \to V(\mu)\otimes V(\lambda)$, and $I(\lambda_j)(j = 1, \ldots, n)$ are the unit matrices denoted by the colours λ_j.

For the convenience of the following discussion we list some relations given in [12] as follows:

$$E(\lambda,\mu)=m^{-1}(G(\lambda,\mu)+G^{-1}(\mu,\lambda))-I(\lambda,\mu) \tag{3.2}$$

$$G(\lambda,\mu)G(\mu,\lambda)=((m+l^{-1})G(\lambda,\mu)+l^{-1}G^{-1}(\mu,\lambda))I(\mu,\lambda)-(1+ml^{-1})I \tag{3.3}$$

$$G_{12}(\lambda,\mu)G_{23}(\lambda,\nu)G_{12}(\mu,\nu)=G_{23}(\mu,\nu)G_{12}(\lambda,\nu)G_{23}(\lambda,\mu) \tag{3.4}$$

$$E_{12}(\lambda,\mu)E_{23}(\lambda,\nu)G_{12}(\mu,\nu)=E_{12}(\lambda,\mu)G_{23}^{-1}(\nu,\lambda)I_{12}(\mu,\nu) \tag{3.5}$$

$$G_{23}(\mu,\nu)E_{12}(\lambda,\nu)E_{23}(\lambda,\mu)=I_{23}(\mu,\nu)G_{12}^{-1}(\nu,\lambda)E_{23}(\lambda,\mu) \tag{3.6}$$

$$E_{12}(\lambda,\mu)E_{23}(\lambda,\nu)G_{12}^{-1}(\mu,\nu)=E_{12}(\lambda,\mu)G_{23}(\lambda,\nu)I_{12}(\mu,\nu) \tag{3.7}$$

$$G_{23}^{-1}(\nu,\mu)E_{12}(\lambda,\nu)E_{23}(\lambda,\mu)=I_{23}(\mu,\nu)G_{12}(\lambda,\nu)E_{23}(\lambda,\mu) \tag{3.8}$$

where $I(\lambda,\mu)$ still satisfies equation (2.2) ($\check{R}(\lambda,\mu)\to G(\lambda,\mu)$).

Substituting equation (3.2) into equations (3.5)-(3.8) and using equations (3.3), (3.4) and (2.2), we obtain

$$F_1(G)-m(F_3(G)+l^{-1}F_4(G)+(m+l^{-1})F_5(G))=0 \tag{3.9}$$

and

$$F_2(G)+m(F_3(G)-(m+l)F_4(G)-lF_5(G))=0. \tag{3.10}$$

Setting

$$\begin{aligned} &G(\lambda,\mu)=(\Lambda_1\Lambda_2)^{-1/2}\check{R}(\lambda,\mu) \qquad m=(\Lambda_1\Lambda_2)^{-1/2}(\Lambda_1+\Lambda_2)\\ &l^{-1}=(\Lambda_1\Lambda_2)^{-1/2}\Lambda_3\end{aligned} \tag{3.11}$$

we find that equation (3.3) is in accordance with equation (2.8). In this case equations (3.9) and (3.10) lead to

$$\Lambda_3^{-1}F_1(\check{R})-p_1F_2(\check{R})-p_2(F_3(\check{R})-p_3^{-1}F_4(\check{R})+p_3F_5(\check{R}))=0 \tag{3.12}$$

where p_j ($j=1,2,3$) are given by

$$p_1=\Lambda_1 \qquad p_2=(\Lambda_2\Lambda_3)^{-1}(\Lambda_1+\Lambda_2)(\Lambda_2+\Lambda_3) \qquad p_3=\Lambda_2 \tag{3.13}$$

or

$$p_1=\Lambda_2 \qquad p_2=(\Lambda_1\Lambda_3)^{-1}(\Lambda_1+\Lambda_2)(\Lambda_1+\Lambda_3) \qquad p_3=\Lambda_1. \tag{3.14}$$

Since equations (3.12) and (3.13) are the same as equation (2.23), so we have proved the correctness of the formula (2.21). Equations (3.12) and (3.14) show that there exists another solution of the YBE (1.3). This solution can be given by exchanging $\Lambda_1\leftrightarrow\Lambda_2$ from the formula (2.21).

4. Examples

4.1. Fundamental representation of A_l

In terms of the weight conservation, the $\check{R}(\lambda,\mu)$-matrices have been calculated in [11]. One solution is

$$\check{R}_I(\lambda,\mu)=\begin{bmatrix} q & & & \\ & & \alpha(\lambda) & \\ & \alpha^{-1}(\mu) & wg(\lambda)g^{-1}(\mu) & \\ & & & q\alpha(\lambda)\alpha^{-1}(\mu)\end{bmatrix} \tag{4.1}$$

where $w = q - q^{-1}$ and $\alpha(\lambda)$, $g(\lambda)$ are arbitrary functions of λ. The eigenvalues of $\check{R}_{\mathrm{I}}(\lambda, \lambda)$ are given by

$$\Lambda_1 = q \qquad \Lambda_2 = -q^{-1}. \tag{4.2}$$

From equations (2.2) and (2.8) we find that

$$\alpha(\lambda, \mu) = \Lambda_1 \Lambda_2 \qquad \beta(\lambda, \mu) = \Lambda_1 + \Lambda_2 \tag{4.3}$$

and

$$I_1(\lambda, \mu) = \begin{bmatrix} 1 & & & \\ & g^{-1}(\lambda) g(\mu) \alpha(\lambda) \alpha^{-1}(\mu) & & \\ & & g(\lambda) g^{-1}(\mu) & \\ & & & \alpha(\lambda) \alpha^{-1}(\mu) \end{bmatrix}. \tag{4.4}$$

Then we have

$$\check{R}(\lambda, \mu; x) = f(\lambda, \mu) \begin{bmatrix} q - q^{-1}x & & & \\ & xwg^{-1}(\lambda) g(\mu) \alpha(\lambda) \alpha^{-1}(\mu) & \alpha(\lambda)(1-x) & \\ & \alpha(\mu)(1-x) & wg(\lambda) g^{-1}(\mu) & \\ & & & \alpha(\lambda) \alpha^{-1}(\mu)(q - q^{-1}) \end{bmatrix}. \tag{4.5}$$

Another solution has the form:

$$\check{R}_{\mathrm{II}}(\lambda, \mu; x) = \begin{bmatrix} q & & & \\ & & X(\lambda) & \\ & Y(\mu) & W(\lambda, \mu) & \\ & & & -q^{-1} X(\lambda) Y(\mu) \end{bmatrix} \tag{4.6}$$

where $X(\lambda)$, $Y(\lambda)$ are arbitrary functions of λ and $W(\lambda, \mu)$ satisfies the relation

$$W(\lambda, \mu) W(\mu, \nu) = (q - q^{-1} X(\mu) Y(\mu)) W(\lambda, \nu). \tag{4.7}$$

The eigenvalues of the $\check{R}(\lambda, \lambda)$ are given by

$$\Lambda_1 = q \qquad \Lambda_2 = -q^{-1} X(\lambda) Y(\lambda) \tag{4.8}$$

in which Λ_2 depends on λ. By solving equations (2.2) and (2.8) we find two solutions of $\alpha(\lambda, \mu)$, $I(\lambda, \mu)$ as follows

$$\alpha(\lambda, \nu) = X(\lambda) Y(\mu)$$

$$I_{\mathrm{II}}^{(1)}(\lambda, \mu) = \beta^{-1}(\lambda, \mu) \begin{bmatrix} A(\lambda, \mu) & & & \\ & T(\lambda, \mu) W(\mu, \lambda) & X(\lambda) Y^{-1}(\lambda)(Y(\lambda) - Y(\mu)) & \\ & X^{-1}(\mu) Y(\mu)(X(\mu) - X(\lambda)) & W(\lambda, \mu) & \\ & & & B(\lambda, \mu) \end{bmatrix} \tag{4.9}$$

where

$$T(\lambda, \mu) = X(\lambda) X^{-1}(\mu) Y^{-1}(\lambda) Y(\mu) \qquad A(\lambda, \mu) = q - q^{-1} X(\lambda) Y(\mu)$$

$$B(\lambda, \mu) = T(\lambda, \mu)(q - q^{-1} X(\mu) Y(\lambda))$$

and

$$\alpha(\lambda,\mu)=X(\mu)Y(\lambda)$$

$$I_{11}^{(2)}(\lambda,\mu)=\beta^{-1}(\lambda,\mu)\begin{bmatrix} q-q^{-1}X(\mu)Y(\lambda) & & & \\ & W(\mu,\lambda) & X(\lambda)-X(\mu) & \\ & Y(\mu)-Y(\lambda) & W(\lambda,\mu) & \\ & & & q-q^{-1}X(\lambda)Y(\mu) \end{bmatrix} \tag{4.10}$$

where $\beta(\lambda,\mu)$ is determined by

$$\beta(\lambda,\mu)\beta(\mu,\lambda)=(q-q^{-1}\alpha(\lambda,\mu))(q-q^{-1}\alpha(\mu,\lambda)).$$

The correspondent $\check{R}(\lambda,\mu;x)$ have the forms

$$\check{R}_{11}^{(1)}(\lambda,\mu;x)=f(\lambda,\mu)\begin{bmatrix} A(\lambda,\mu;x) & & & \\ & T(\lambda,\mu)W(\mu,\lambda) & C(\lambda,\mu;x) & \\ & W(\lambda,\mu) & D(\lambda,\mu;x) & \\ & & & B(\lambda,\mu;x) \end{bmatrix} \tag{4.11}$$

with

$$A(\lambda,\mu;x)=q-q^{-1}X(\lambda)Y(\mu)x$$

$$B(\lambda,\mu;x)=T(\lambda,\mu)(qx-q^{-1}X(\mu)Y(\lambda))$$

$$C(\lambda,\mu;x)=X(\lambda)Y^{-1}(\lambda)(Y(\lambda)-Y(\mu)x)$$

$$D(\lambda,\mu;x)=X^{-1}(\mu)Y(\mu)(X(\mu)-X(\lambda)x)$$

and

$$\check{R}_{11}^{(2)}(\lambda,\mu;x)=f(\lambda,\mu)\begin{bmatrix} q-q^{-1}X(\mu)Y(\lambda)x & & & \\ & W(\mu,\lambda)x & X(\lambda)-X(\mu)x & \\ & Y(\mu)-Y(\lambda)x & W(\lambda,\mu) & \\ & & & q-q^{-1}X(\lambda)Y(\mu) \end{bmatrix} \tag{4.12}$$

4.2. Fundamental representations of $A_n (n>1)$

In this case the $\check{R}(\lambda,\mu)$-matrix has the form [12]

$$\check{R}(\lambda,\mu)=\sum_a u_a(\lambda,\mu)e_{aa}\otimes e_{aa}+\sum_{a<b} w^{(a,b)}(\lambda,\mu)e_a\otimes e_{bb} + \sum_{a\neq b} p^{(a,b)}(\lambda,\mu)e_{ab}\otimes e_{ba} \tag{4.13}$$

where

$$(e_{ab})_{cd}=\delta_{ac}\delta_{bd},\ a \qquad b\in\left[-\frac{N-1}{2},-\frac{N-1}{2},\dots,\frac{N-1}{2}\right] \qquad N=n+1.$$

The coefficients $u_a(\lambda,\mu)$, $w^{(a,b)}(\lambda,\mu)$ and $p^{(a,b)}(\lambda,\mu)$ are given by

$$\begin{aligned} u_a(\lambda,\mu)&=u_a g_a(\lambda)g_a^{-1}(\mu) \qquad p^{(a,b)}(\lambda,\mu)=g_b(\lambda)g_a^{-1}(\mu) \\ w^{(a,b)}(\lambda,\mu)&=(q-q^{-1})g_b(\lambda)g_b^{-1}(\mu)h_a(\lambda)h_b^{-1}(\lambda)h_a^{-1}(\mu)h_b(\mu) \end{aligned} \tag{4.14}$$

with $u_a = q$ or $u_a \in q, -q^{-1}$. It is not difficult to find that the eigenvalues of the $\check{R}(\lambda, \lambda)$, $\alpha(\lambda, \mu)$ and $\beta(\lambda, \mu)$ are the same as those given by equations (4.2) and (4.3). The corresponding $I(\lambda, \mu)$ is obtained as follows

$$I(\lambda, \mu) = \sum_a g_a(\lambda) g_a^{-1}(\mu) e_{aa} \otimes e_{aa} + \sum_{a \neq b} (q - q^{-1})^{-1} w^{(a,b)}(\lambda, \mu) e_{aa} \otimes e_{bb}. \tag{4.15}$$

Then we have

$$\begin{aligned}\check{R}(\lambda, \mu; x) = f(\lambda, \mu) \Big\{ & \sum_a (u_a - u_a^{-1} x) g_a(\lambda) g_a^{-1}(\mu) e_{aa} \otimes e_{aa} \\ & + \Big(\sum_{a<b} + x \sum_{a>b} \Big) w^{(a,b)}(\lambda, \mu) e_{aa} \otimes e_{bb} \\ & + (1 - x) \sum_{a \neq b} g_b(\lambda) g_a^{-1}(\mu) e_{ab} \otimes e_{ba} \Big\}. \end{aligned} \tag{4.16}$$

4.3. Fundamental representations of B_n, C_n, D_n

$$\Lambda_1 = q \qquad \Lambda_2 = -q^{-1} \qquad \Lambda_3 = \begin{cases} q^{-2n} & \text{for } B_n \\ q^{-2n-1} & \text{for } C_n \\ q^{-2n+1} & \text{for } D_n. \end{cases} \tag{4.17}$$

The $\check{R}(\lambda, \mu)$ can be computed in terms of the weight conservation [12]. We rewrite them in the form

$$\begin{aligned}\check{R}(\lambda, \mu) = & \sum_{a \neq 0} u_a(\lambda, \mu) e_{aa} \otimes e_{aa} + \sum_{a<b a \neq -b} w^{(a,b)}(\lambda, \mu) e_{aa} \otimes e_{bb} \\ & + \sum_{a \neq \pm} p^{(a,b)} e_{ab} \otimes e_{ba} + \sum_{a,b} q^{a,b}(\lambda, \mu) e_{a-b} e_{-ab} \end{aligned} \tag{4.18}$$

where

$$a, b \in \left[-\frac{N-1}{2}, -\frac{N-1}{2} + 1, \ldots, \frac{N-1}{2} \right] \qquad N = 2n+1, 2n$$

for B_n, C_n (or D_n) respectively. The coefficients $u_a(\lambda, \mu), \ldots, q^{(a,b)}(\lambda, \mu)$ are given by

$$u_a(\lambda, \mu) = q g_a(\lambda) g_a^{-1}(\mu) \qquad w^{(a,b)}(\lambda, \mu) = (q - q^{-1}) g_b(\lambda) g_b^{-1}(\mu)$$

$$p^{(a,b)}(\lambda, \mu) = g_b(\lambda) g_a^{-1}(\mu)$$

$$q^{(a,b)}(\lambda, \mu) = \begin{cases} q^{-1} g_{-a}(\lambda) g_a^{-1}(\mu) & (a = b \neq 0) \\ g_0(\lambda) g_0^{-1}(\mu) & (a = b = 0) \\ (q - q^{-1})(-\varepsilon_a \varepsilon_b q^{\bar{a}\bar{b}} + \delta_{a-b}) g_{-a}(\lambda) g_b^{-1}(\mu) & (a < b) \\ 0 & (a > b). \end{cases} \tag{4.19}$$

Here $\varepsilon_a = 1$ for B_n, D_n, $\varepsilon_a = 1 (a < 0)$, $-1 (a > 0)$ for C_n, and $\tilde{a} = a + \frac{1}{2}(a < 0)$, $a (a = 0)$, $a - \frac{1}{2}(a > o)$ for B_n, D_n, $\tilde{a} = a - \frac{1}{2}(a < 0)$, $a + \frac{1}{2}(a > 0)$, for C_n. The corresponding $\check{R}^{-1}(\mu, \lambda)$ can be given by

$$\begin{aligned}\check{R}^{-1}(\mu, \lambda) = & q^{-1} \sum_{a \neq 0} g_a(\lambda) g_a^{-1}(\mu) e_{aa} \otimes e_{aa} - (q - q^{-1}) \sum_{a > b, a \neq -b} g_b(\lambda) g_b^{-1}(\mu) e_{aa} \otimes e_{bb} \\ & + \sum_{a \neq \pm b} g_b(\lambda) g_a^{-1}(\mu) e_{ab} \otimes e_{ba} + \sum_{a,b} \bar{q}^{(a,b)}(\mu, \lambda) e_{a-b} \otimes e_{-ab} \end{aligned} \tag{4.20}$$

where

$$\bar{q}^{(a,b)}(\mu,\lambda)=\begin{cases} qg_{-a}(\lambda)g_a^{-1}(\mu) & (a=b\neq 0)\\ g_0(\lambda)g_0^{-1}(\mu) & (a=b=0)\\ (q-q^{-1})(\varepsilon_a\varepsilon_b q^{\bar{a}-\bar{b}}-\delta_{a-b})g_{-a}(\lambda)g_b^{-1}(\mu) & (a>b)\\ 0 & (a<b). \end{cases} \tag{4.21}$$

In [12] we have proved that these $\check{R}(\lambda,\mu)$-matrices admit the BW algebraic structure with colour. Therefore the $\check{R}(\lambda,\mu;x)$ can be obtained from equation (2.22) as follows

$$\check{R}_a(\lambda,\mu;x)=\Lambda_1 x(x-1)\check{R}^{-1}(\mu,\lambda)+Q_2 I(\lambda,\mu)-\Lambda_3(x-1)\check{R}(\lambda,\mu) \tag{4.22}$$

and

$$\check{R}_b(\lambda,\mu;x)=\Lambda_2 x(x-1)\check{R}^{-1}(\mu,\lambda)+Q_1 I(\lambda,\mu)-\Lambda_3(x-1)\check{R}(\lambda,\mu) \tag{4.23}$$

where

$$\begin{aligned} I(\lambda,\mu)&=\sum_{a,b} g_b(\lambda)g_b^{-1}(\mu)e_{aa}\otimes e_{bb}\\ Q_k&=\Lambda_k^{-1}\Lambda_3(\Lambda_1+\Lambda_2)(\Lambda_k+\Lambda_3) \qquad (k=1,2). \end{aligned} \tag{2.24}$$

5. Conclusion and discussion

In this paper, starting from the given $\check{R}(\lambda,\mu)$ in appropriate form, we present the prescription for Yang–Baxterization of coloured $\check{R}$-matrix to generate trigonometic solutions of the YBE (1.3). The structure of the $\check{R}(\lambda,\mu)$ depends on reduction relations satisfied by the $\check{R}(\lambda,\mu)$. The reduction relation should be reduced to the characteristic equation for $\check{R}(\lambda,\lambda)$ when $\lambda=\mu$. For simplicity, we only consider the cases of the $\check{R}(\lambda,\lambda)$ having two and three distinct eigenvalues. When three distinct eigenvalues are independent of λ, we find that this prescription is related to the BW algebra with colour and two solutions are generated from the $\check{R}(\lambda,\mu)$ satisfying the BW algebra with colour. If three distinct eigenvalues depend on λ, the discussion given in section 2 is not correct. We need to construct a new form for $\check{R}(\lambda,\mu;x)$ instead of equation (2.18).

We would like to point out that the $\check{R}(\lambda,\mu)$ can be represented by 'coloured projector' $P_k(\lambda,\mu)$ as follows:

$$\check{R}(\lambda,\mu)=\sum_{k=1}^{m}\Lambda_k(\lambda,\mu)P_k(\lambda,\mu) \tag{5.1}$$

and

$$\check{R}^{-1}(\lambda,\mu)=\sum_{k=1}^{m}\Lambda_k^{-1}(\lambda,\mu)P_k(\mu,\lambda) \tag{5.2}$$

where $\Lambda_k(\lambda,\mu)$ are the determining coefficients satisfying $\Lambda_k(\lambda,\mu)=\Lambda_k(\mu,\lambda)$. When $\lambda=\mu$, $\Lambda_k(\lambda,\mu)$ will be reduced to the distinct eigenvalues of $\check{R}(\lambda,\lambda)$. $P_k(\lambda,\mu)$ are defined by

$$P_k(\lambda,\mu)P_l(\mu,\lambda)=\delta_{kl}P_k(\lambda,\mu)I(\mu,\lambda)=\delta_{kl}I(\lambda,\mu)P_k(\mu,\lambda) \tag{5.3}$$

where $I(\lambda,\mu)$ satisfies

$$I(\lambda,\mu)I(\mu,\lambda)=I(\lambda,\lambda)=I.$$

Considering the relation $\Sigma_{k=1}^{m} P_k(\lambda, \mu) = I(\lambda, \mu)$, we have

$$\prod_{k=1}^{m} (\check{R}(\lambda, \mu) I(\mu, \lambda) - \Lambda_k(\lambda, \mu)) = 0. \tag{5.4}$$

In terms of equations (5.1), (5.3) and (5.4), we can express the $P_k(\lambda, \mu)$ as

$$P_k(\lambda, \mu) = \left[\prod_{l \neq k}^{m} \frac{(\check{R}(\lambda, \mu) I(\mu, \lambda) - \Lambda_l(\lambda, \mu))}{(\Lambda_k(\lambda, \mu) - \Lambda_l(\lambda, \mu))} \right] I(\lambda, \mu). \tag{5.5}$$

By generalizing the method given in [8] we can construct

$$\check{R}(\lambda, \mu; x) = \sum_{k=1}^{m} \Lambda_k(\lambda, \mu; x) P_k(\lambda, \mu) \tag{5.6}$$

where

$$\Lambda_k(\lambda, \mu) = \sum_{n=0}^{m-1} a_k^{(n)}(\lambda, \mu) x^n.$$

According to this consideration, we can also write the $\check{R}(\lambda, \mu; x)$ in the form of equations (2.10) and (2.18) for $m = 2$ and 3, respectively.

However, it is too difficult to determine $\Lambda_k(\lambda, \mu)$ for the given $\check{R}(\lambda, \mu)$ because $\Lambda_k(\lambda, \mu)$ are not the eigenvalues of the $\check{R}(\lambda, \mu)$. So we start to construct, from the reduction relations the forms for $\check{R}(\lambda, \mu; x)$ in section 2.

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